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# STUDY MATERIALS

Class:  B.Sc (GE)

Batch : Semester II

Topic: Algebra

Subtopic: Functions

Singnature: Satya Kumar Das

Date : 06.04.2020

Time: 07 am

Duretion: 40 mins

Class attended :03 (three)

# FUNCTIONS

**Definition:**

A function or mapping (Defined as $f:X\rightarrow Y$) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.

Function 'f' is a relation on X and Y such that for each $x\in X$, there exists a unique $y\in Y$ such that $(x,y)\in f$. 'x' is called pre-image and 'y' is called image of function f.

**Injective / One-to-one function:**

A function $f:A\rightarrow B$ is injective or one-to-one function if for each pair of distinct elements of A , their f-images are distinct.

This means a function f is injective if $a_1\neq a_2$ implies $f(a_1)\neq f(a_2)$, where $a_1,a_2 \in A$.

**Example**

$f:N\rightarrow N, f(x)=5x$ is injective.

$f:N\rightarrow N, f(x)=x^2$ is injective.

$f:R\rightarrow R, f(x)=x^2$ is not injective as $(-x)^2=x^2$

**Surjective / Onto function:**

A function $f:A\rightarrow B$ is surjective (onto) if the image of f equals its range. Equivalently, for every $b\in B$, there exists some $a\in A$ such that $f(a)=b$. This means that for any y in B, there exists some x in A such that $y=f(x)$.

**Example**

$f:N\rightarrow N$, $f(x)=x$ is surjective.

$f:R\rightarrow R, f(x)=x^2$ is not surjective since we cannot find a real number whose square is negative.

**Bijective / One-to-one Correspondent:**

A function $f:A\rightarrow B$ is bijective or one-to-one correspondent if and only if f is both injective and surjective.

**Example**

Prove that a function $f:R \to R$ defined by $f(x)=2x–3$ is a bijective function.

Proof:

We have to prove this function is both injective and surjective.

If $f(x_1)=f(x_2)$, then $2x_1–3=2x_2–3$ and it implies that $x_1=x_2$.

Hence, *f* is injective.

Here, 2x–3=y

So, x=(y+5)/3 which belongs to R and f(x)=y.

Hence, *f* is surjective.

Since *f* is both surjective and injective, we can say f is bijective.

**Identity function.**

**Definition :** The identity function $i_A$ on the set A is defined by:

$i_A : A \to A$, $i_A(x) = x$ where $x \in A$.

**Invertible function:**

**Definition.** Let f : A → B be a bijection. Then the inverse function of f, $f^{-1} : B \to A$ is defined elementwise by: $f^{-1}(b)$ is the unique element $a \in A$ such that f(a) = b. We say that f is invertible function.

**Example**

A Function $f:Z \to Z$, $f(x)=x+5$, is invertible since it has the inverse function $f^{-1}:Z \to Z$, $f^{-1}(x)=x-5$.

A Function $f:Z \to Z$, $f(x)=x^2$ is not invertiable since this is not one-to-one as $(-x)^2=x^2$.

**Composition of Functions:**

Two functions $f:A \to B$ and $g:B \to C$ can be composed to give a composition gof. This is a function from A to C defined by (gof)(x)=g(f(x))

**Example**

Let f(x)=x+2 and g(x)=2x+1, find (fog)(x) and (gof)(x).

Solution:

*(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3*

*(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+5*

*Hence, (fog)(x)≠(gof)(x)*

**Theorem1.** If f : X → Y is a one-to-one correspondence, then $f^{-1}$ : Y → X is a one-to-one correspondence.

Proof. To prove this, we just apply the definition of bijection, namely, we need to show that $f^{-1}$ is an injection, and a surjection. Let us start with injection.

$f^{-1}$ is an injection: we have to prove that if $f^{-1}(y_1) = f^{-1}(y_2)$, then $y_1 = y_2$. All right, then $f^{-1}(y_1) = f^{-1}(y_2) = x$ for some x in X. But $f^{-1}(y_1) = x$ means that $y_1 = f(x)$, and $f^{-1}(y_2) = x$ means that $y_2 = f(x)$, by definition of the inverse of function. But this shows that $y_1 = y_2$.

$f^{-1}$ is an surjection: by definition, we need to prove that any x ∈ X has a preimage, that is, there exists y such that $f^{-1}(y) = x$. Because f is a bijection, there is some y such that y = f(x), therefore $x = f^{-1}(y)$ .

Therefore $f^{-1}$ : Y → X is a one-to-one correspondence. (Proved)

**Theorem 2.** Let f : X → Y and g : Y → Z be two injective functions. Then g ◦ f is also injective.

Proof. What we need to do is check the injectivity of a function, so we do this as usual: we have to show that $g \circ f(x_1) = g \circ f(x_2)$ implies $x_1 = x_2$. Now both f and g are injective. So let us start. We have $g \circ f(x_1) = g \circ f(x_2)$ or equivalently $g(f(x_1)) = g(f(x_2))$. But we know that g is injective, so this implies $f(x_1) = f(x_2)$. Next we use that f is injective, thus $x_1 = x_2$. Hence the proved.

**Theorem 3.** Let f : X → Y and g : Y → Z be two surjective functions. Then g ◦ f is also surjective.

Proof. The codomain of $g \circ f$ is $Z$, therefore we need to show that every $z \in Z$ has a preimage $x$, namely that there always exists an $x$ such that $g \circ f(x) = z$. Now f and g are both surjective. Since g is surjective, we know there exists $y \in Y$ such that $g(y) = z$. Now again, since f is surjective, we know there exists $x \in X$ such that $f(x) = y$. Therefore there exist x, y such that $z = g(y) = g(f(x))$. Hence the proved.

**Home Works:**

**H1:** $f(x) = 2x+3$ and $g(x) = x^2$ find *(fog)(x)* and *(gof)(x)*.

**H2:** $f(x) = \sqrt{x}$ and $g(x) = x^2$ find *(fog)(x)* and *(gof)(x)*.

The End