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# STUDY MATERIALS

Class: B.Sc (GE2)

Batch : Semester II (GE2)

Subject: Algebra (Unit-2)

Topic: Equivalance Relations

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# RELATION

Relations may exist between objects of the same set or between objects of two or more sets.

**Definition:**

A binary relation R from set X to Y (written as xRy or R(x,y)) is a subset of the Cartesian product X×Y.

A binary relation R on a single set A is a subset of A×A.

**Domain and Range**

If there are two sets A and B, and relation R have order pair (x, y), then −

The domain of R, Dom(R), is the set { x|(x,y) ∈R for some y in B}

The range of R, Ran(R), is the set { y|(x,y) ∈R for some x in A}

**Examples**

Let, A={1,2,9} and B={1,3,7}

1. If relation R is defined as R={(1,1),(3,3)} then
   Dom(R) = {1,3},
   Ran(R)={1,3}
2. If relation R is defined as R={(1,3),(1,7),(2,3),(2,7)} then
   Dom(R) = {1,2},
   Ran(R)={3,7}

**Types of Relations:**

**I. Reflexive:**

A relation R on set A is called Reflexive if ∀a∈A is related to a.

(i.e aRa holds ∀a∈A)

**Example:** The relation R={(a,a),(b,b)} on set X={a,b} is reflexive.

**II. Symmetric:**

A relation R on set A is called Symmetric if xRy implies yRx, where x, y∈A.

**Example:** The relation R={(1,2),(2,1),(3,2),(2,3)} on set A={1,2,3} is symmetric.

**III. Transitive:**

A relation R on set A is called Transitive if xRy and yRz implies xRz, where x,y,z∈A.

**Example** :The relation R={(1,2),(2,3),(1,3)} on set A={1,2,3} is transitive.

**IV: Equivalence Relation:**

A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive.

**Example:** The relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is an equivalence relation since it is reflexive, symmetric, and transitive.

**Q1:** Let R be the relation on the set ℝ real numbers defined by xRy iff $x - y$ is an integer. Prove that R is an equivalence relation on ℝ.

Proof.

**I. Reflexive:** Suppose $x \in \mathbb{R}$. Then $x - x = 0$, which is an integer. Thus, xRx $\forall x \in \mathbb{R}$.

**II. Symmetric:** Suppose $x, y \in \mathbb{R}$ and xRy. Then $x - y$ is an integer. Since $y - x = -(x - y)$, $y - x$ is also an integer. Thus, yRx .

**III. Transitive:** Suppose $x, y \in \mathbb{R}$, xRy and yRz. Then $x - y$ and $y - z$ are integers. Thus, the sum $(x - y) + (y - z) = x - z$ is also an integer, and so xRz.

Thus, R is an equivalence relation on ℝ.

**Equivalence Classes:**

**Definition:** Let R be an equivalence relation on A and let a $\epsilon$ A. The set [a] = {x $\epsilon$ A :aRx} is called the equivalence class of a. The element in the bracket in the above notation is called the representative of the equivalence class.

**Theorem 1. Let R be an equivalence relation on a set A. Then the following are equivalent:**

**(1) aRb**

**(2) [a] = [b]**

**(3) [a] ∩ [b] ≠ ∅**

**Proof.**

1 → 2. Suppose a, b $\epsilon$ A and aRb. We must show that [a] = [b].
Suppose x $\epsilon$ [a]. Then, by definition of [a], aRx. Since R is symmetric and aRb, bRa. Since R is transitive and we have both bRa and aRx, bRx. Thus, x $\epsilon$ [b].
Suppose x $\epsilon$ [b]. Then bRx. Since aRb and R is transitive, aRx. Thus, x $\epsilon$ [a].
We have now shown that x $\epsilon$ [a] if and only if x $\epsilon$ [b]. Thus, [a] = [b].

2→3. Suppose a, b $\epsilon$ A and [a] = [b]. Then [a] ∩ [b] = [a]. Since R is reflexive, aRa; that is a $\epsilon$ [a]. Thus [a] = [a] ∩ [b] ≠ ∅.

3→1. Suppose [a] ∩ [b] ≠ ∅;. Then there is an x $\epsilon$ [a] ∩ [b]. By definition, aRx and bRx. Since R is symmetric, xRb. Since R is transitive and both aRx and xRb, aRb.

**Partition**

**Definition:** A collection P of nonempty subsets of a set A is a partition of A if

(1) S∩S‘ =∅ ; if S and S‘ are in P and S ≠S‘, and

(2) A =∪{S: S $\epsilon$ P}.

**Q2 :** Let m be a positive integer. Prove that the relation $a \equiv b \pmod{m}$, is an equivalence relation on the set of integers.

**Proof.**

**I. Reflexive:** If a is an arbitrary integer, then $a - a = 0 = 0 \cdot m$. Thus $a \equiv a \pmod{m}$.

**II. Symmetric:** If $a \equiv b \pmod{m}$, then $a - b = k \cdot m$ for some integer k. Thus, $b - a = (-k) \cdot m$ is also divisible by m, and so $b \equiv a \pmod{m}$.

**III.Transitive:** Suppose $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then $a - b = k \cdot m$ and $b - c = n \cdot m$ for some integers k and n. Then

$a - c = (a - b) + (b - c) = k \cdot m + n \cdot m = (k + n)m$ is also divisible by m. That is, $a \equiv c \pmod{m}$.

Therefore the relation $a \equiv b \pmod{m}$, is an equivalence relation on the set of integers. (Proved)

**Remark :**

Given a positive integer m, we know that the relation $a \equiv b \pmod{m}$, is an equivalence relation on the set of integers. If we use the Division Algorithm to divide the integer a by the integer m, we get a quotient q and remainder r, $0 \leq r < m$, satisfying the equation $a = mq + r$. Recall that $r \equiv a \pmod{m}$ and that $a \equiv r \pmod{m}$. Thus $[a] = [r]$, and so there are exactly m equivalence classes $[0], [1], \ldots [m - 1]$.

**Home Work:**

**H1:** All these problems concern a set A = {1,2,3,4}.

Relation R1 = { (1,1), (2,2), (3,3), (4,4) }.

Relation R2 = { (1,1), (2,2), (2,3), (3,2) }.

Relation R3 = { (3,2), (2,3) }.

Relation R4 = { (1,1), (1,2), (2,2), (1,3), (3,3), (1,4), (4,4) } .

Relation R5 = { (1,2), (2,3), (1,3), (2,2) }.

Fill the blanks

| **RELATIONS** | **R1** | **R2** | **R3** | **R4** | **R5** |
|---|---|---|---|---|---|
| **Reflexive?** | | | | | |
| **Symmetric?** | | | | | |
| **Transitive?** | | | | | |
| **Equivalence?** | | | | | |